Calculate the limit

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$$\lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\ldots\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right)^n.$$

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Note that $\lim_{n\to\infty}\left(\left(\frac{1}{\sqrt{e}}\prod_{k=1}^{n}\left(1+\frac{k}{n^2}\right)\right)^n\right) = e^{\lim_{n\to\infty} a_n}$, where $a_n := \ln\left(\left(\frac{1}{\sqrt{e}}\prod_{k=1}^{n}\left(1+\frac{k}{n^2}\right)\right)^n\right) = n\left(\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)-\frac{1}{2}\right), n \in \mathbb{N}$.

Since $x-\frac{x^2}{2} < \ln(1+x) < x-\frac{x^2}{2}+\frac{x^3}{3}, \forall x \in (0,1)$ then

$$n\left(\sum_{k=1}^{n}\left(\frac{k}{n^2}-\frac{k^2}{2n^4}\right)-\frac{1}{2}\right) < a_n < n\left(\sum_{k=1}^{n}\left(\frac{k}{n^2}-\frac{k^2}{2n^4}+\frac{k^3}{3n^6}\right)-\frac{1}{2}\right) \Leftrightarrow$$

(*) $$\frac{(4n+1)(n-1)}{12n^2} < a_n < \frac{(4n+1)(n-1)}{12n^2}+\frac{(n+1)^2}{12n^3}.$$

Noting that $\lim_{n\to\infty}\frac{(4n+1)(n-1)}{12n^2} = \frac{1}{3}$ and $\lim_{n\to\infty}\frac{(n+1)^2}{12n^3} = 0$ we obtain (by Squeeze Principle) $\lim_{n\to\infty} a_n = 1/3$ and, therefore

$$\lim_{n\to\infty}\left(\left(\frac{1}{\sqrt{e}}\prod_{k=1}^{n}\left(1+\frac{k}{n^2}\right)\right)^n\right) = e^{1/3}.$$

* $n\left(\sum_{k=1}^{n}\left(\frac{k}{n^2}-\frac{k^2}{2n^4}\right)-\frac{1}{2}\right) = n\left(\sum_{k=1}^{n}\frac{k}{n^2}-\sum_{k=1}^{n}\frac{k^2}{2n^4}-\frac{1}{2}\right) = n\left(\frac{n(n+1)}{2}\cdot\frac{1}{n^2}-\frac{1}{2}-\frac{1}{2n^4}\cdot\frac{n(n+1)(2n+1)}{6}\right) = \frac{1}{2}-\frac{(2n+1)(n+1)}{12n^2} = \frac{(4n+1)(n-1)}{12n^2}$ and $n\cdot\sum_{k=1}^{n}\frac{k^3}{3n^6} = \frac{1}{3n^5}\cdot\frac{n^2(n+1)^2}{4} = \frac{(n+1)^2}{12n^3}$.